ON THE LEVI GRAPH OF POINT-LINE CONFIGURATIONS

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ABSTRACT. We prove that the well-covered dimension of the Levi graph of a point-line configuration with v points, b lines, r lines incident with each point, and every line containing k points is equal to 0, whenever $r > 2$.

1. INTRODUCTION

The concept of the well-covered space of a graph was first introduced in 1998 and 1999 by Caro, Ellingham, Ramey, and Yuster (see [**?**] and [**?**]) as an effort to generalize the study of well-covered graphs. Brown and Nowakowski [**?**], in 2005, continued the study of this object and, among other things, provided several examples of graphs featuring odd behaviors regarding their well-covered space. One of these special situations occurs when the well-covered space of the graph is trivial, i.e. when the graph is *anti-well-covered*. In this work, we prove that almost all Levi graphs of configurations in the family of the so-called (v_r, b_k) configurations (see Definition 3) are anti-well-covered.

We start our exposition by providing the following definitions and previously-known results. Any introductory concepts we do not present here may be found in the books by Bondy & Murty [**?**] and Grünbaum [**?**].

We consider only simple and undirected graphs. A graph will be denoted as $G = (V(G), E(G))$, as is customary, where $V(G)$ is the set of vertices of the graph and $E(G)$ is the set of edges of the graph. We think of $E(G)$ as an irreflexive symmetric relation on $V(G)$. Two vertices of a graph are said to be *adjacent* if they are connected by an edge. An *independent* set of vertices is one in which no two vertices in the set are adjacent. If an independent set, M, of a graph G, is not a proper subset of any other independent set of G, then M is a *maximal independent set* of G.

Definition 1. *Let G be a graph and $\mathbf{F}$ a field.*

(1) *A function $f : V(G) \to \mathbf{F}$ is said to be a weighting of G. If the sum of all weights is constant for all maximal independents sets of G, then the weighting is a well-covered weighting of G.*

(2) *The $\mathbf{F}$-vector space consisting of all well-covered weightings of G is called the well-covered space of G (relative to $\mathbf{F}$).*

(3) *The dimension of this vector space is called the well-covered dimension of G, denoted $wcdim(G, \mathbf{F})$.*

Remark 1. *For some graphs, the characteristic of the field* **F** *makes a difference when calculating the well-covered dimension (see* [**?**] *and* [**?**]*). If* $char(\mathbf{F})$ *does not cause a change in the well-covered dimension, then the well-covered dimension is denoted as* $wcdim(G)$.

In order to calculate the well-covered dimension of a graph, G, one would generally need to find all possible maximal independent sets of G. However, finding all maximal independent sets is not always an easy task, as this is a known NP-complete problem.

Despite the NP-complete nature of this problem, let us assume that we have found all possible maximal independent sets of G. We will denote these maximal independent sets as M_i for $i = 0, 1, \ldots, k-1$. The well-covered weightings of G are determined by solving a system of linear equations that arise from considering all equations of the form $M_0 = M_i$ for $i = 1, \ldots, k-1$. We replace this system with the equivalent homogenous one via standard operations and create an associated matrix A_G. Observe that the dimension of the nullspace of A_G is equal to the dimension of the well-covered space of G. Thus,

$$wcdim(G, \mathbf{F}) = |V(G)| - rank(A_G).$$

We now move onto another component of our work: configurations.

Definition 2. *A (point-line) configuration is a triple* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$*, where* $\mathcal{P}$ *is set of points,* $\mathcal{L}$ *is a set of lines,* $\mathcal{I}$ *is an incidence relation between* $\mathcal{P}$ *and* $\mathcal{L}$*, and that has the following properties:*

(1) *Any two points are incident with at most one line;*
(2) *Any two lines are incident with at most one point.*

Next, there is some notation for configurations that needs to be set, as well as specific parameters that need to be established for the main result of this work.

Definition 3. *We define a* (v_r, b_k) *configuration as a configuration such that*

(1) $|\mathcal{P}| = v$*, and* $v \geq 4$.
(2) $|\mathcal{L}| = b$*, and* $b \geq 4$.
(3) *There are exactly* k *points incident with each line, and* $k \geq 2$.
(4) *There are exactly* r *lines incident with each point, and* $r \geq 2$.

When $v = b$ and $r = k$, the configuration will be denoted by (v_r).

Example 1. *Several well-known geometric structures fall into the category of* (v_r, b_k) *configurations. For instance:*

(1) *A projective plane of order* q *is a* $(q^2 + q + 1_{(q+1)})$ *configuration, where* q *is the power of a prime. See Figure* **??** *for a representation of* $PG(2,3) = (13_4)$.
(2) *The Pappus configuration is a* (9_3) *configuration, and the Desargues configuration is a* (10_3) *configuration.*
(3) $PG(n, q)$ *is a* $\left(\dfrac{q^{n+1} - 1}{q - 1}_{(q+1)}, \dfrac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}_{(q^2+q+1)} \right)$ *configuration, where* q *is the power of a prime.*
(4) *A generalized quadrangle* $G(s, t)$ *is a* $((1+s)(st+1)_{(1+s)}, (1+t)(st+1)_{(1+t)})$ *configuration.*

The reader is referred to the book by Batten [?] for more information about these important geometric objects.

Finally, we define the object, Levi graphs, that will connect configurations and graphs.

Definition 4. *The Levi graph of a (v_r, b_k)-configuration $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the bipartite graph G with $V(G) = \mathcal{P} \cup \mathcal{L}$ and $E(G) = \mathcal{I}$. That is, $p \in \mathcal{P}$ is adjacent to $\ell \in \mathcal{L}$ if and only if $p\mathcal{I}\ell$. We will denote this graph $Levi_{(v_r, b_k)}$.*
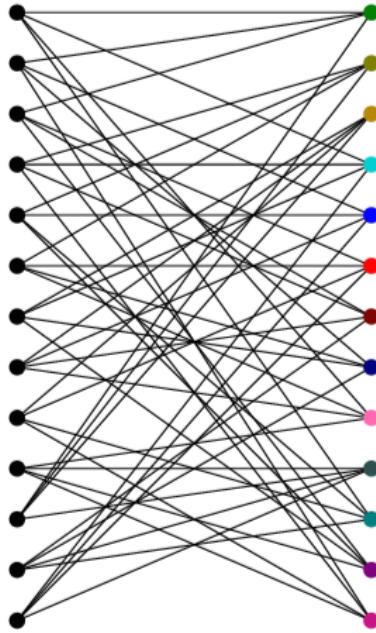*Note that $\mathcal{P}$ and $\mathcal{L}$ are independent sets - the partite sets - in G.*

FIGURE 1. $(13_4) = PG(2,3)$ and $Levi_{(13_4)}$

Our main result, which will be proven in the following section, combines all these objects as follows:

Theorem 1. *If r is a positive integer greater than 2, then $wcdim(Levi_{(v_r, b_k)}) = 0$.*

We would like to remark that Theorem ?? says is that almost all Levi graphs of (v_r, b_k) configurations are anti-well-covered.

2. THE WELL-COVERED DIMENSION OF $Levi_{(v_r, b_k)}$

We will prove Theorem ?? by first proving a technical lemma, that introduces a family of maximal independent sets that will show to be useful later on.

Lemma 1. *A Levi graph of a configuration (v_r, b_k), where $r > 2$, has at least $v + b + 2$ maximal independent sets.*

Proof. Let P be a fixed point in (v_r, b_k). We consider the set, M_P, of vertices of $Levi_{(v_r, b_k)}$ given by P and all the lines not incident to P. This is an independent set of $Levi_{(v_r, b_k)}$ because there is no incidence between vertices in the set. Moreover, note that if we included another point-vertex to M_P, then that vertex would be adjacent to one of the line-vertices in M_P (because of condition (2) in Definition 2, and the fact that $r > 2$). Also, if another line-vertex were to be added to M_P, then this line would have to be incident with P. It follows that M_P is a maximal independent set of $Levi_{(v_r, b_k)}$.
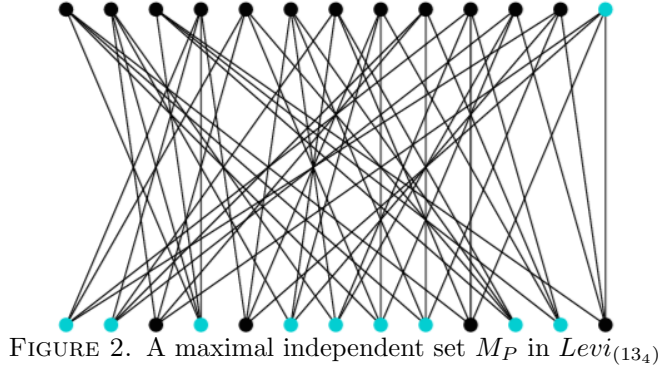

FIGURE 2. A maximal independent set M_P in $Levi_{(13_4)}$

Repeating this construction for all v points in (v_r, b_k), we get v distinct maximal independent sets of $Levi_{(v_r, b_k)}$.

We will now construct another b distinct maximal independent sets of $Levi_{(v_r, b_k)}$. We start by fixing a line ℓ in (v_r, b_k) and then any two distinct points $P_1, P_2 \in \ell$ (recall that $k \geq 2$). We consider the set, M_{P_1, P_2} of vertices of $Levi_{(v_r, b_k)}$ given by P_1, P_2 and all the lines not incident to either of these points. Note that this forms an independent set since adjacency in $Levi_{(v_r, b_k)}$ only occurs if incidence occurs in (v_r, b_k). If we try to add in another vertex-point to M_{P_1, P_2}, since $r > 2$, then this point will be incident to one of the lines not through P_1 or P_2 and will therefore be adjacent to the vertex-lines in M_{P_1, P_2}. If we try to add another vertex-line to M_{P_1, P_2}, then this line will be incident to one or both P_1 and P_2. Therefore, M_{P_1, P_2} is a maximal independent of $Levi_{(v_r, b_k)}$.
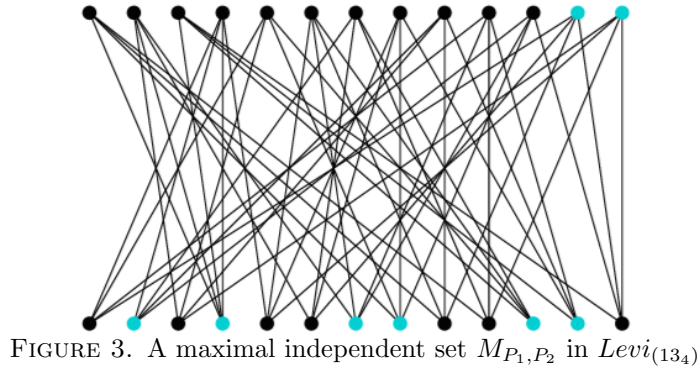

FIGURE 3. A maximal independent set M_{P_1, P_2} in $Levi_{(13_4)}$

Repeating this construction for all b lines in (v_r, b_k) (it does not matter what pair of points one picks on any given line), we get b distinct maximal independent sets of $Levi_{(v_r, b_k)}$.

Finally, note that the set of all point-vertices in $Levi_{(v_r, b_k)}$ is a maximal independent set as well as the set of all line-vertices in $Levi_{(v_r, b_k)}$. Hence, we have constructed $v + b + 2$ distinct maximal independent sets in $Levi_{(v_r, b_k)}$. □

Next, we proceed to prove our main result.

Proof of Theorem **??**. We denote by $\mathbf{F}$ the field of scalars of the well-covered space of $G = Levi_{(v_r, b_k)}$, where $r > 2$. Let A_G be the associated matrix of G, and note that A_G has $v + b$ columns. In order to prove that A_G has $v + b$ linearly independent rows we will consider the $v + b + 2$ maximal independent sets in Lemma **??**.

We create the first v rows of A_G by equating the weight of each of the maximal independent sets M_P to the weight of the maximal independent set consisting of all the lines of G. After subtracting we obtain v equations of the form

$$(1) \qquad f(P) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) = 0$$

where each ℓ_i is incident with P. It follows that, after organizing the columns of A_G by putting point-vertices first and then line-vertices, the 'first' v rows of A_G are

$$\left[\begin{array}{cc} I_v & -C \end{array} \right]$$

where C is the incidence matrix of $Levi_{(v_r, b_k)}$.

In order to obtain the next b rows of A_G we will consider maximal independent sets of the form $M_{P,Q}$. For any given line ℓ of (v_r, b_k), we choose (any) two points on it. We will denote these two points as P_1 and P_2. We then consider the maximal independent set M_{P_1, P_2} and equate its weight to the weight of the maximal independent set M_{P_1}. After subtracting we obtain an equation of the form

$$(2) \qquad f(P_2) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) + f(\ell) = 0$$

where each ℓ_i is incident with P_2.

Note that subtracting Equation **??** (with $P = P_2$) from Equation **??** yields $f(\ell) = 0$. Since ℓ was arbitrary, we get $f(\ell) = 0$ for every line in (v_r, b_k). It follows that since subtracting equations is just a different way to describe row operations in A_G, we get that the 'first' $v + b$ rows of A_G (after a few row operations) are

$$\left[\begin{array}{cc} I_v & -C \\ \mathbf{0} & I_b \end{array} \right]$$

Note that addition and subtraction were the only two (row) operations needed to obtain the matrix above. Hence, the first $v + b$ rows of A_G do not change depending on the characteristic of $\mathbf{F}$.

Since the determinant of the matrix above is non-zero, the rank of A_G is maximal, and thus $wcdim(Levi_{(v_r, b_k)}) = 0$. □

3. POSSIBLE GENERALIZATIONS

In this section, we study possible generalizations of Theorem **??**. This will be done by providing a few results and by introducing objects for which this theorem could be extended to. We begin by proving that Theorem **??** cannot be extended

to configurations having exactly two lines being incident with every point. This will be done by an example that considers (v_2) configurations.

We first notice that a (v_2) configuration is a disjoint union of polygons/cycles. This is convenient because disjoint unions of graphs behave well with respect to the well-covered dimension. In fact, Lemma 5 in [?] says

$$wcdim(G \cup H) = wcdim(G) + wcdim(H),$$

where $\cup$ stands for disjoint union.

Since we know that $Levi_{C_n} = C_{2n}$, we get the following lemma.

Lemma 2. *Let $\mathcal{C}$ be a (v_2) configuration. Then,*

$$\mathcal{C} = \bigcup_{i=1}^{t} C_{n_i},$$

where $n_i > 2$, for all $1 \le i \le t$. Moreover,

$$wcdim(Levi_{\mathcal{C}}) = \sum_{i=1}^{t} wcdim(C_{2n_i})$$

Finally, we notice that Theorem 5 in [?] implies

$$wcdim(C_{2n}) = \begin{cases} 2 & \text{if } n = 3 \\ 0 & \text{if } n \ge 4 \end{cases}$$

Next is an immediate corollary of Theorem 5 in [?] and Lemma **??**.

Corollary 1. *$wcdim(Levi_{\mathcal{C}})$ is even, for all (v_2) configurations $\mathcal{C}$.*
Moreover, for every $n \in \mathbb{N}$, there is a (v_2) configuration, $\mathcal{C}_n$, such that

$$wcdim(Levi_{\mathcal{C}_n}) = 2n$$

In particular, the sequence $\{wcdim(Levi_{\mathcal{C}_n})\}_{n=1}^{\infty}$ is unbounded.

We conclude that Theorem **??** cannot be expanded to the case $r = 2$. However, it is still an open problem to find the well-covered dimension of all Levi graphs of (v_2, b_k) configurations.

Of course, the study of the well-covered dimension of Levi graphs of configurations not of the form (v_r, b_k) is also an interesting open problem.

Block designs are another family of objects that could be studied to attempt a generalization of Theorem **??**. These objects can be much less 'geometric' than (v_r, b_k) configurations, given that they are obtained after relaxing items (3) and (4) in Definition 2. In order to be more precise, we provide the following definition.

Definition 5. *Let $\lambda, t \ge 1$. A t-(v, k, λ) design (or t-design), is an incidence structure of points and blocks with the following properties:*

    (1) *there are v points;*
    (2) *each block is incident with k points;*
    (3) *any t points are incident with λ common blocks.*

It is easy to see that a 1-(v, k, λ) design is a (v_λ, b_k) configuration, where $b = v\lambda/k$. Moreover, a 2-$(v, k, 1)$ design is a configuration in which every pair of points are 'collinear.' For $t > 1$ and $\lambda > 1$, the obvious definition of the Levi graph of a t-design would yield a multigraph. This apparent setback is not so much of a problem since having one edge or multiple edges between two vertices would mean

the same when looking for maximal independent sets. We claim that the ideas used to prove Theorem **??** can be generalized to be applicable to block designs.

Finally, in this work, we studied the well-covered space of the Levi graph of a (v_r, b_k) configuration. We propose, as an interesting open problem, the study of configurations via understanding the well-covered spaces of their collinearity graphs (in which points in a configuration are defined as vertices, and adjacency occurs if and only if the points are collinear). The third author is currently working on a particular case of this problem: generalized quadrangles.

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