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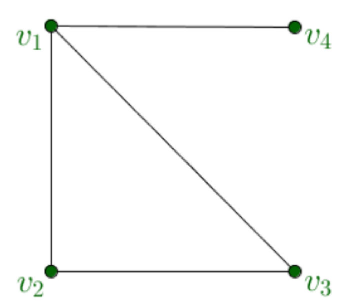
# The Well-Covered Dimension of Point-Line Configurations

## Well-Covered Dimension

An *independent set* is a set of vertices in a graph in which no two vertices in the set are connected by an edge. A *maximal independent set* is an independent set that is not a subset of any other independent set.

Example: The maximal independent sets of this graph are $\{v_1\}, \{v_2, v_4\}, \{v_3, v_4\}$.




A *weighting* of a graph, $G$, is a function $f$:V$(G) \to \mathbf{F}$ that assigns a value from the field $\mathbf{F}$ to each vertex of $G$. A *well-covered weighting* is a weighting such that $\sum_{x \in M} f(x)$ is constant for every $MIS$ of $G$.

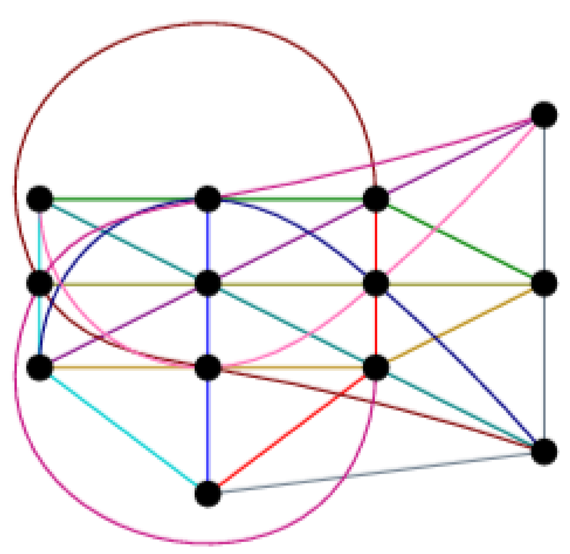
The set of all well-covered weightings of a graph $G$ over a field $\mathbf{F}$ is a vector space, and the dimension of this vector space is called the *well-covered dimension* of the graph, denoted wcdim$(G, \mathbf{F})$.

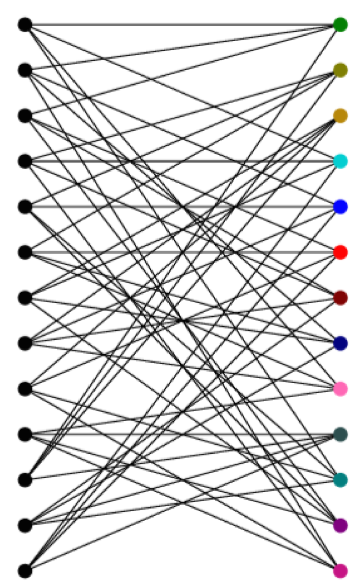
## Point-line configurations and Levi Graphs

We define a $(v_r, b_k)$ *configuration* as a point-line configuration such that

1. There are exactly $k$ points incident to every line, and k $\geq 2$.
2. There are exactly $r$ lines incident with each point, and $r \geq 2$.
3. There are exactly $v$ points in $(v_r, b_k)$, and $v \geq 4$.
4. There are exactly $b$ lines in $(v_r, b_k)$, and $b \geq 4$.

For the *Levi graph* of a $(v_r, b_k)$ configuration, the graph, denoted $Levi_{(v_r, b_k)}$, is a bipartite graph with vertices defined as the sets of points and lines, and edges are the incidence relations between the points and lines. That is, two vertices are connected iff a point and line are incident in the configuration.





## Creating the Associated Matrix

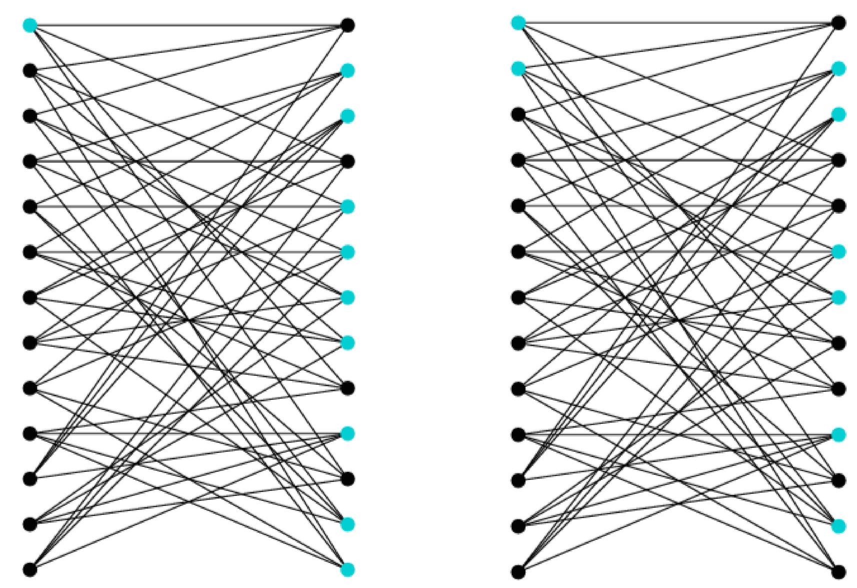
$A$, *the associated matrix*, is an $m \times n$ matrix in which $m$ is the number of maximal independent sets and $n$=|V$(G)$|, the cardinality of the vertex set. $A$ is formed by calculating the well-covered weighting of $Levi_{(v_r, b_k)}$.

Consider the first $v + b$ rows of our associated matrix, i.e. the system of equations relating the weightings of the vertices in $Levi_{(v_r, b_k)}$. We shall use specific maximal independent sets which we have found and sum them in such a way so that the first $v + b$ rows take on the following shape,

$$\begin{bmatrix} I_v & -\mathrm{C} \\ \mathbf{0} & \mathrm{I}_b \end{bmatrix}$$

where I is the Identity matrix, C represents the negative of the incidence matrix of $Levi_{(v_r, b_k)}$, and $\mathbf{0}$ represents the zero matrix.

To create the first $v + b$ rows, maximal independent sets of the following types were selected.



Lemma: The first $v + b$ rows of the associated matrix are linearly independent.

## Theorem on the Well-Covered Dimension of $Levi_{(v_r, b_k)}$

Theorem: If $r$ is a positive integer greater than 2, then

$$wcdim(Levi_{(v_r, b_k)}) = 0.$$

It is important to note for the proof that the well-covered dimension of a graph is the nullity of the associated matrix since the system of equations is homogeneous. Since we know the first $v + b$ rows are linearly independent, the rank is $v + b$, and the nullity is 0.

## Examples

The following objects are a sample of the several well known geometric structures that fall into the category of $(v_r, b_k)$ configurations.

1. A projective plane of order $q$ is a $(q^2 + q + 1_{(q+1)})$ configuration, where $q$ is a power of a prime.
2. The Pappus configuration is a $(9_3)$ configuration, and the Desargues configuration is a $(10_3)$ configuration.
3. PG(n,q) is a $(\frac{q^{n+1}-1}{q-1}_{(q+1)}, \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}_{(q^2+q+1)})$ configuration where $q$ is a power of a prime
4. A generalized quadrangle $G(s, t)$ is a $((1 + s)(st + 1)_{(1+s)}, (1 + t)(st + 1)_{(1+t)})$ configuration.

## Restriction on $r$

Notice a $(v_2)$ configuration is a disjoint union of polygons/circles. The Levi graph of these configurations are in fact cyclic graphs, which have well-covered dimensions other than 0. This work regarding the well-covered dimension of cyclic graphs can be found in [**?**]. Because of this work, we offer this following corollary:

Corollary: $wcdim(Levi_{\mathcal{C}})$ is even, for all $(v_2)$ configurations $\mathcal{C}$.
Moreover, for every $n \in \mathbb{N}$, there is a $(v_2)$ configuration, $\mathcal{C}_n$, such that

$$wcdim(Levi_{\mathcal{C}_n}) = 2n$$

In particular, the sequence $\{wcdim(Levi_{\mathcal{C}_n})\}_{n=1}^{\infty}$ is unbounded.
Due to the fact that our theorem cannot be expanded to the case $r = 2$, it is still an open problem to find the well-covered dimension of all Levi graphs of $(v_2, b_k)$ configurations as well as Levi graphs not of the form $(v_r, b_k)$.

## Bibliography

[1] I. Birnbaum, R. McDonald, M. Kuneli, K. Urabe, and O. Vega. The well-covered dimension of products of graphs. *Discussiones Mathematicae Graph Theory* 34, 811–827. 2014

## Acknowledgements

*We gratefully acknowledge the support from the NSF Grant #DMS-1156273, and the REU program at Fresno State for the summer of 2014.*