Math 197: Senior Thesis

Chromatic, Orbital Chromatic Polynomials and their Roots

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Introduction

This thesis is an exploration of chromatic polynomials, orbital chromatic polynomials and attempting to prove the following conjecture,

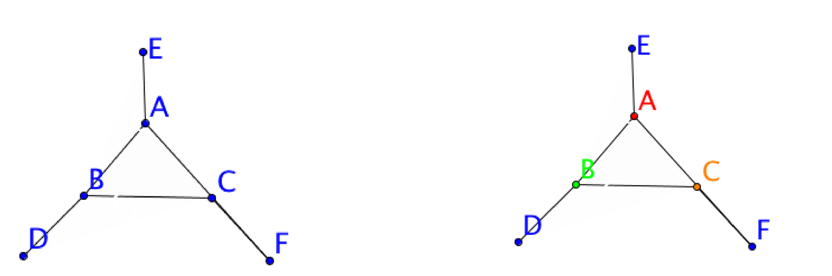
Conjecture 1 *(?) For $N > 0$, there exists a graph Γ and an automorphism group G of Γ for which $OP_{\Gamma,G}(x)$ has a root at least N larger than the largest real root of $P_\Gamma(x)$.*

Graph Theory Basics

To understand the topic of my research let us consider some definitions from graph theory,

Definition 2 *A graph, $\Gamma = (V, E)$, is a set V, of vertices and a set of edges where each edge is a 2 element subset of V.*

Definition 3 *A proper k-coloring of a graph Γ, is a function, $c : V \to \{1, ..., k\}$ such that $c(u) \neq c(v)$ for any edge $\{u, v\}$.*



(a) A graph Γ (b) A proper coloring of Γ

Figure 1: A graph and a proper coloring of the graph

Chromatic Polynomial

Definition 4 *(?) The chromatic polynomial, $P_\Gamma(k)$ of a graph Γ and a positive integer k is the number of proper k-colorings of a graph.*

Example: We shall now compute the chromatic polynomial for the graph Γ in Figure 1. Let us begin by noting that the vertices A, B and C are all adjacent to one another, so there is $x(x-1)(x-2)$ ways to color these vertices. Now note that vertices D, E and F are only adjacent to one vertex each so there are $x - 1$ possible colors for each of these vertices. We then find the chromatic polynomial of the graph Γ in Figure 1 is, $P_\Gamma(x) = x(x-1)^4(x-2)$.

Orbital Chromatic Polynomials

Definition 5 *(?) For a graph Γ, a group G of automorphisms of Γ, and a positive integer k, the orbital chromatic polynomial $OP_{\Gamma,G}(k)$, is the number of unique k-colorings of Γ. Where two colorings are equivalent if one can be obtained from another by an automorphism in G.*

A closed form for $OP_{\Gamma,G}(x)$ can also be written as,

Theorem 6 *The orbital chromatic polynomial of a graph Γ with respect to a group G is,*

$$OP_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x).$$

Let us now find $OP_{\Gamma,G}(x)$ for some Γ and G,

Example: Let us compute the orbital chromatic polynomial, $OP_{\Gamma,G}(x)$, of a path graph Γ on three vertices and the group $G = \{0°, 180°\}$, where the group elements are rotations of the graph. From Theorem 6 we know how to compute $OP_{\Gamma,G}(x)$ from the chromatic polynomials of $\Gamma/0°$ and $\Gamma/180°$. These two graphs are shown in Figure 2.

Orbital Chromatic Polynomials Cont.

(a) $\Gamma/0°$ (b) $\Gamma/180°$

Figure 2: A graph Γ and $\Gamma/180°$

We find the chromatic polynomial of $\Gamma/0°$ is $P_{\Gamma/0°}(x) = x(x-1)^2$ and the chromatic polynomail of $\Gamma/180°$ to be $P_{\Gamma/180°}(x) = x(x-1)$. So we find the orbital chromatic polynomial of the graph Γ and group G to be, $OP_{\Gamma,G}(x) = \frac{1}{2}(x(x-1)^2 + x(x-1))$.

Results

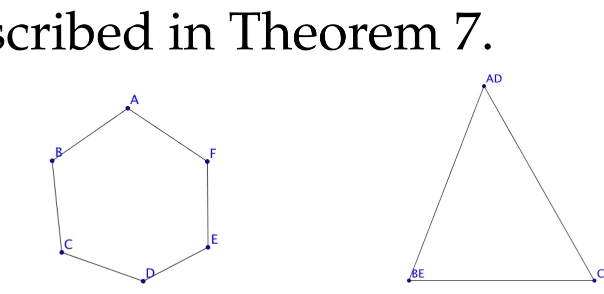
I have written the following theorem which describes the characteristics of graphs whose existence would prove Conjecture 1.

Theorem 7 *Let Γ be a graph and G be a group of automorphisms. Suppose the following hold:*

1. *The largest real root of $P_\Gamma(x)$ is m.*
2. *There is some $g \in G$ for which Γ/g has less vertices than any of the graphs $\{\Gamma/h : h \in G, h \neq g\}$.*
3. *For the g in (2) Γ/g is a complete graph on j vertices where $m < j$.*
4. *There exists some $x_0 > m$ such that $P_{\Gamma/g}(x_0) < 0$.*

Then one can construct from Γ and G, a graph Γ' and a group of automorphims of Γ', called G', such that the $OP_{\Gamma',G'}(x)$ has a real root that is at least j - 1-m larger than the largest real root of $P_\Gamma(x)$.

In Figure 3 we see an example of a graph Γ, a cycle graph on 6 vertices, and a group $G = \{0°, 180°\}$ of automorphisms of Γ that have the characteristics described in Theorem 7.



(a) A graph Γ (b) $\Gamma/180°$

Figure 3: A graph with characteristics described in Theorem 7

The largest root of $P_\Gamma(x)$ is 1 and $\Gamma/180°$ is a complete graph on 3 vertices. By Theorem 7 we know the orbital chromatic polynomial will have a root that is 1 larger than the largest real chromatic root of $P_\Gamma(x)$. The goal now is to find a family of graphs which have the characteristic described above, in order to prove Conjecture 1.

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